On the Stability of Medial Axis of a Union of Disks in the Plane

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Abstract

We show that the medial axis of union of disks in the plane is stable provided that the topology is preserved and every disk meets the boundary in a single arc. If the second condition is removed, the medial axis is no longer stable, but if pruned using any of four significance measures (circumradius, erosion thickness, object angle or potential residue) it remains stable.

1 Introduction

The medial axis of a shape has proven to be one of the most useful tools in computational geometry. One of its most important properties is that it represents the full topology of the shape [5]. For most shape representations there are no known algorithms to calculate the medial axis exactly. The only context where it can be calculated efficiently for sizable datasets is when the shape is a union of balls [2]. Because of this, most practical algorithms that utilize the medial axis do so for shapes represented as a finite union of balls.

For general shapes, it is well known that the medial axis is unstable under small perturbations. One approach to stability has been to deal with subsets of the medial axis, such as the λ-medial axis, which can be shown to represent the topology of the shape and be stable under small perturbations of the shape [4].

In this paper, we consider a different approach; we restrict our family of shapes to be a union of disks in the plane and study how the medial axis can change when disks move and change size. The one restriction that we will use thoughout is that the topology of the shape will not be allowed to change.

First we will show that if we add the additional assumption that each disk meets the boundary of the shape in at most one arc then the medial axis is stable, i.e. the medial axis changes continuously in the Hausdorff metric.

The second result applies to several significance measures on the medial axis. If we allow disks to change their intersection patterns with the boundary but not the topology of the shape, the medial axis can change substantially but the significance measures are still stable. This implies that is we were to truncate the medial axis using these significance measures, similar to the λ-medial axis, then the truncated medial axis change continuously in the Hausdorff metric.

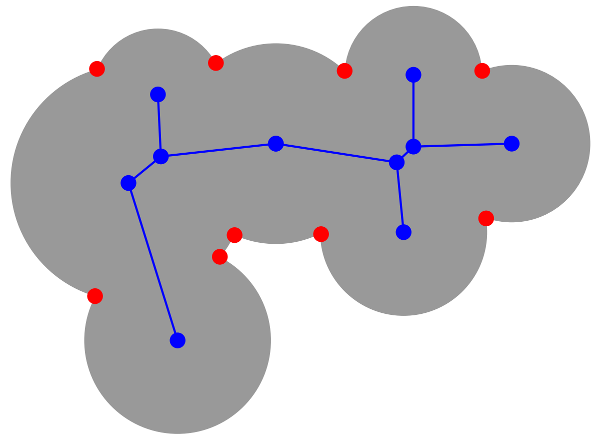

Figure 1: The medial axis of a union of disks. The singular points and vertices of the medial axis are marked.

2 Medial Axis of a Union of Disks

Both Attali and Montanvert [3] and Amenta and Kolluri [2] have characterized the medial axis of a union of balls. A simple modification of either yields the following characterization of the medial axis of a union of disks in the plane. It utilizes the Voronoi diagram of the *singular points*, the points of intersection of the boundary of two or more disks. These points are precisely where the boundary of the union of disks fails to be differentiable.

Lemma 1 *Suppose that $X \subset \mathbb{R}^2$ is a union of disks that is also a surface with boundary then the medial axis of X is a graph where*

- *Vertices are either vertices of the Voronoi diagram of the singular points of X or centers of disks with two or more singular points on their boundaries.*
- *Edges between vertices connect two vertices and are equidistant from two singular points with no closer singular point.*

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Definition 2 *A union of disks, $X \subset \mathbb{R}^2$, is* generic *if*

- *The disks have positive radii and distinct centers.*
- *No three centers are co-linear.*
- *X is a manifold with boundary.*
- *No singular point lies on the boundary of three or more disks.*
- *Any disks contained in X that has four or more singular points on its boundary is one of the disks comprising the union.*

Otherwise, the union of disks is called degenerate.

For generic unions of disks, the vertices of its medial axis are either centers of disks or Voronoi vertices with exactly three adjacent edges. See Figure 1 for an example.

3 Configurations of Unions of Disks

We will distinguish between the set of disks and the shape their union forms. A single disk in $\mathbb{R}^2$ can be represented as a tuple (x, y, r) storing the disk's center and radius (which is required to be non-negative). The set of configurations of n disks in $\mathbb{R}^2$ will be denoted $\mathcal{C}_n$ which is a subset of $\mathbb{R}^{3n}$. For a configuration $\alpha \in \mathcal{C}_n$, we will denote the corresponding shape $X_\alpha \subset \mathbb{R}^2$ as the union of the disks specified by α.

Consider a path γ in $\mathcal{C}_n$ and the corresponding unions of disks. As the disks move around and change sizes there are two ways the medial axis can change. The first is that singular points could appear or disappear. The second is the Voronoi diagram of the singular points changes. In the study of dynamic Voronoi diagrams, it has been shown that if points move continuously in the plane then the Voronoi diagram changes continuously in the Hausdorff metric and it combinatorics change when more than three points lie on a circle simultaneously [1].

Each possible change to the medial axis corresponds to a particular degenerate configuration of the disks. We will show that any path in the configuration space can be infinitesimally perturbed to hit these degeneracies in isolated points and only one at a time. By studying what happens at each of these degeneracies we will be able to understand the stability of the medial axis.

The degenerate configurations correspond to each of the conditions in definition 2. Each can be described by a set of polynomial constraints, shown below. Assume that $\{(x_i, y_i)\}$ are the centers of the disks and $\{r_i\}$ are their radii. We will use (s_j, t_j) to represent the coordinates of a singular point.

Zero radius disk Some fixed disk has zero radius.

$$r_i = 0 \text{ for some } i$$

Centers coinciding Two fixed disks have identical centers.

$$(x_i, y_i) = (x_j, y_j) \text{ for some } i \neq j$$

Co-linear centers Three of more centers of circles are co-linear.

$$\begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{vmatrix} = 1 \text{ for some distinct } i, j, k$$

Co-tangent disks The boundaries of two specified disks are tangent to each other.

$$(x_i - x_j)^2 + (y_i - y_j)^2 = (r_i \pm r_j)^2 \text{ for some } i, j$$

Triple point A singular point is the intersection of the boundary of three of the disks.

$$(x_i - x)^2 + (y_i - x)^2 = r_i^2, \forall i \in I$$

where $|I| = 3$ and (x, y) are the coordinates of the triple point

Co-circular singular points There are four specified singular points that lie on a circle that is not the boundary of one of the disks in the configuration.

$$\begin{aligned} (x_i - s_j)^2 + (y_i - t_j)^2 &= r_i^2 \\ (x - s_j)^2 + (y - t_j)^2 &= r^2 \end{aligned}$$

Where each pair (i, j) determines which circles the singular points lie on and (x, y) is the center of the circle of radius r the singular points lie on

Since solutions to polynomial systems have zero measure you can immediately show the following.

Proposition 3 *For any configuration $\alpha \in \mathcal{C}_n$ and $\epsilon > 0$ there exists a generic configuration $\beta \in \mathcal{C}_n$ such that $d(\alpha, \beta) < \epsilon$.*

The results in this paper utilize two main techniques. The first, allows us to perform infinitesimal perturbations of paths in the configuration space to ones that are generic almost everywhere along their length. The medial axis changes smoothly between these few degenerate configurations. By analyzing what happens to the medial axis at each of these degeneracies we will be able to see exactly how the medial axis can change and what happens to several significance measures.

Proposition 4 *For any path $\gamma : [0, 1] \to \mathcal{C}_n$ with endpoints being generic configurations and $\epsilon > 0$ there exists a path $\gamma_\epsilon[0, 1] \to \mathcal{C}_n$ such that*

- $\gamma_\epsilon(0) = \gamma(0), \gamma_\epsilon(1) = \gamma(1)$
- $\text{length}(\gamma_\epsilon) < \text{length}(\gamma) + \epsilon$
- $d_H(\gamma_\epsilon, \gamma) < \epsilon$
- *For all but finitely many values of t, $\gamma(t)$ is a generic configuration and at those values the configuration has exactly one degeneracy.*

If we define $\mathcal{G}_n \subset \mathcal{C}_n$ to be all configurations that have at most one degeneracy, then the proposition implies that

$$d_{\mathcal{G}_n}(\alpha, \beta) = d_{\mathcal{C}_n}(\alpha, \beta)$$

So instead of having to worry about configurations with multiple degeneracy, we can assume that we only have configurations that are either generic or have exactly one degeneracy. The major steps in the proof are:

1. Perturb the path so that no disk has zero radius.
2. The set of configurations where two centers coincide is a co-dimension 2 linear subspace of the configuration space. Any path can be perturbed off of all of these subspaces.
3. The configurations that do not satisfy the other genericity conditions form an algebraic variety in a possibly extended set of coordinates. It can be shown that the singular points of these varieties and the intersections of these varieties are all co-dimension 2 or higher and can be avoided.
4. Finally, the path can be perturbed to intersect each of the manifold regions of these varieties transversely.

4 Changes to the Medial Axis at Degenerate Configurations

Figure 2 shows the changes to the medial axis at each type of degenerate configuration. The figures are slightly before, at and slightly after the degenerate configuration. Only configurations where the topology of the union of disks is unchanged and where the medial axis combinatorics are changed. At all other degeneracies the medial axis moves continuously.

The first of the three cases deals with co-tangent disks. Notice that this tangency occurs with one disk contained in another. If they were adjacent then either the point would be in the interior of another disk or a change in topology would occur. In the first case there is not change in the combinatorics of the medial axis and the second violates our assumptions. When one disk is contained in another, immediately after the disk emerges a new edge is created. This edge goes between the centers of the two disks.

In the second case in the figure, a disk emerges from the union of disks passing through a singular point in the boundary. Similar to the previous case a new edge appears instantaneously. The edge goes between the center of an emerging disk and a newly formed Voronoi vertex. In the degenerate configuration this Voronoi vertex lies on a pre-existing edge that is adjacent to the singular point the disk is passing through.

In the final case, the singular points change positions but do not appear or disappear. This case is covered in the literature on dynamic Voronoi diagrams [1].

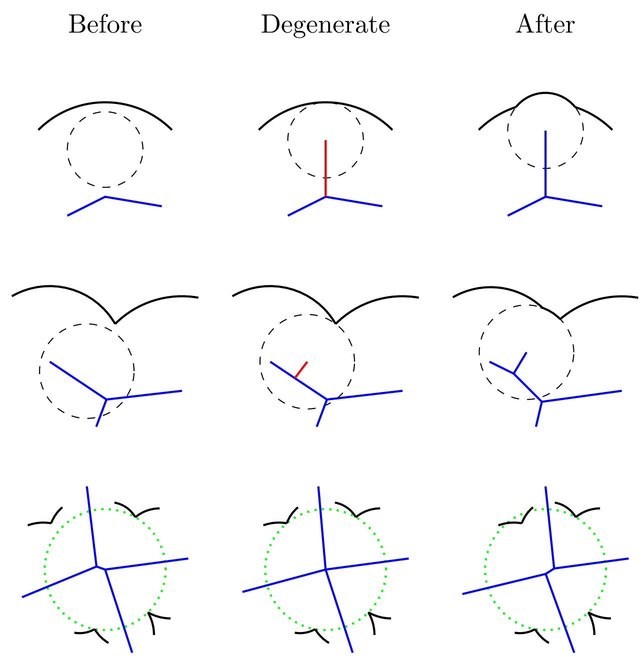

Figure 2: The changes in the medial axis for (a) cotangent disks, (b) triples points and (c) four circular singular points. The relevant portion of the medial axis is shown in blue. The red edges represent edges of the medial axis that appear instantaneously after the moving circle meets the boundary.

One edge of the medial axis collapses to a point and a different edge emerges. Throughout this combinatorial change, the medial axis changes continuously in the Hausdorff metric.

If we are willing to change the shape slightly, then we can assume each hits the boundary in a single arc. If this property is maintained through a perturbation singular points never appear or disappear and the only changes in the combinatorics of the medial axis are edges shrinking to a point and the reverse. Away from these changes in combinatorics, the disks, singular points and vertices of the medial axis all move continuously. This shows that, in the Hausdorff metric, the union of disks change continuously.

Theorem 5 *If $B \subset \mathcal{G}_n$ consists of configurations of disks where every disk meets the boundary of the shape in at most a single arc, then the medial axis of X_α varies continuously in the Hausdorff metric as α varies continuously over configurations in B.*

It is worth noting that the analogous result is not true in 3-dimensions.

5 Stability of Significance Measures

We will examine several significance measures on the medial axis. These are all real valued functions, with larger values corresponding to more significant regions. These measures we study are circumradius, object angle [9], potential residue [7] and erosion thickness [6, 8]. Note that the λ-medial axis uses circumradius as its significance measure to prune the medial axis.

Consider a simply connected shape $X \subset \mathbb{R}^2$ and a point x on the medial axis M. The significance measures can be defined as follows:

Circumradius $R(x)$ is the minimum radius of a disk in X containing the two closest points on the boundary to x.

Object angle $OA(x)$ is the angle between the vectors from x to the two closest points on ∂X to x.

Potential residue $PR(x)$ is the distance on ∂X between the two closest points on ∂X to x.

Erosion thickness We choose a definition equivalent to the one in [6], $ET(x)$ is defined as

$$\max\{\min\{d_M(x,l)+D(l), d_M(x,l')+D(l')\}-D(x)\}$$

where $d_M(x,y)$ is distance measured on the medial axis, $D(x) = d(x, \partial X)$ and the maximum is taken over all leaves l, l' such that x is on the path between the two leaves.

As a shape is perturbed, we will show that each of these significance measures changes continuously as the vertices move. In particular, when new spurs are created in the medial axis, see Figure 2, all of these significance measures are zero on the newly created points. We will adapt the Hausdorff metric to measure the differences between functions with different domains.

Definition 6 *Consider two sets $X_1, X_2 \subset \mathbb{R}^2$ and functions $f_i : X_i \to \mathbb{R}$, the extended Hausdorff distance $d_{EH}((X_1, f_1), (X_2, f_2))$ is defined as $d_H(Y_1, Y_2)$ where $Y_i = \{(x, f_i(x)) \mid x \in X_i\} \cup \big(\mathbb{R}^2 \times \{0\}\big)$.*

Notice that two spaces are considered close in this extended Hausdorff distance if every point has a small function value or it is close to a point on the other set that has a similar function value.

We can examine what happens to each of these significant measure as the union of disks changes under the assumption that there is no change in the topology of the union of disks. There are two ways an edge could disappear: shrinking continuously to a point and disappearing when two disks become cotangent. An edge appears in the reverse of either of these processes. See Figure 2 for examples. Consider a point x on one of these edges that appear instantaneously. Immediately after the edge appears, there are two singular points that are closest to the edge. As you get closer to the time where the edge appears, these singular points merge. In the limit cirucumradius, object angle and potential residue are all identically zero on the edge. Erosion thickness is also zero on the edge at the time the edge appears since the shortest path from any point on the edge to the boundary goes through a leaf of the medial axis.

Away from these degenerate configurations the singular points and the medial axis all move continuously. It is easily shown that all four significance measures also change continuously with the changes in the shape. This yields the following theorem.

Theorem 7 *If f is either circumradius, object angle, potential residue or erosion thickness then $\mathcal{MA}(X_\alpha)$ varies continuously as α moves continuously in $\mathcal{G}_n$.*

Corollary 8 *For every $t \geq 0$,*

$$M_{\alpha,t} = \{x \in \mathcal{MA}(X_\alpha) \mid f(x) \geq t\}$$

varies continuously in the Hausdorff metric as α moves continuously in $\mathcal{G}_n$.

This corollary implies that if you truncate the medial axis using any of these significance measures then it changes continuously as the union of disks are perturbed. An example of this is shown in Figure 3 using erosion thickness. It is worth noting that for arbitrary truncation values, only erosion thickness is known to preserve topology. This corollary is already known for sufficiently small thresholds for circumradius since in this case $M_{\alpha,t}$ is the λ-medial axis.

Future Work

There are several natural extensions that are worth considering. These include considering more general shapes and extending to three dimensions. Another nice project would be building kinetic data structure to efficiently track the changes in the medial axis as the balls move around.

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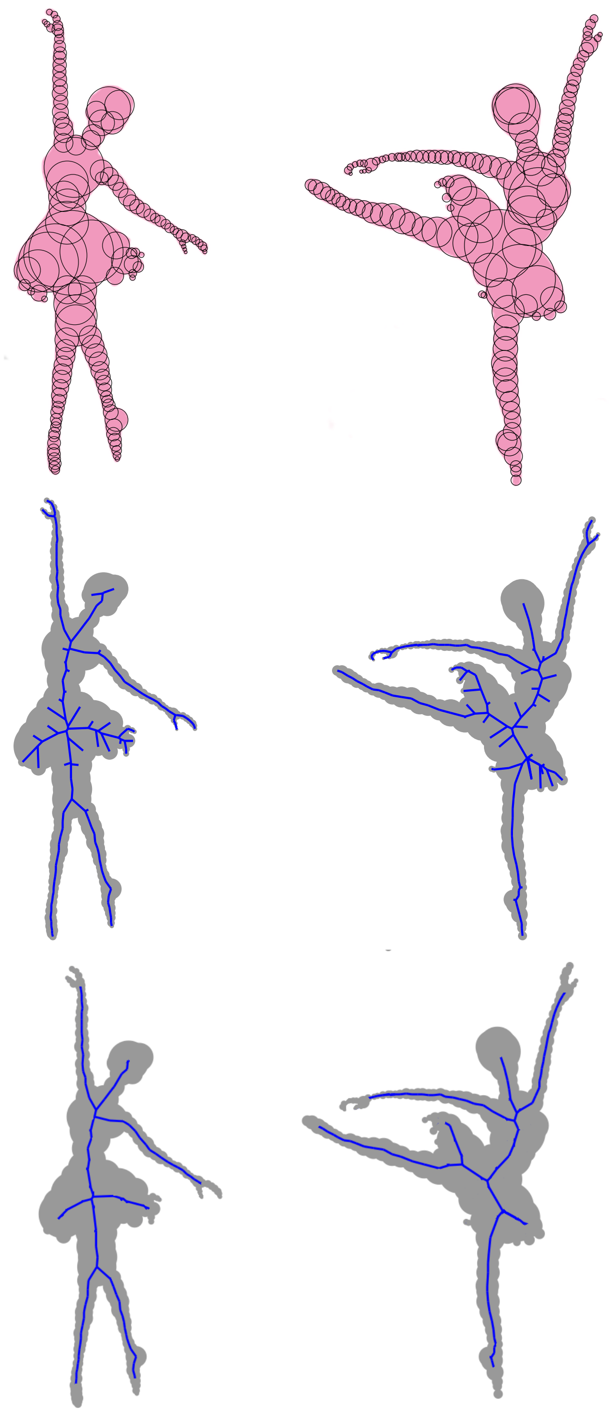

Figure 3: Two poses of a ballerina. Top row: the union of disks representing the shapes. Middle row: the medial axes. Bottom row: the medial axes pruned using erosion thickness. As the dancer moves these pruned medial axes transform continuously (in the Hausdorff metric) from one to the other.

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