ALGORITHMS FOR ESSENTIAL SURFACES IN 3-MANIFOLDS

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Abstract. In this paper we outline several algorithms to find essential surfaces in 3-dimensional manifolds. In particular, the classical decomposition theorems of 3-manifolds (Kneser-Milnor connected sum decomposition and the JSJ decomposition) are defined by splitting along families of disjoint essential spheres and tori. We give algorithms to find such surfaces, using normal and almost normal surface theory and the technique of crushing triangulations. These algorithms have running time $O(p(t)3^t)$, where t is the number of tetrahedra in any given initial one-vertex triangulation of the manifold and $p(t)$ is some polynomial in t. A special instance of these ideas gives a new algorithm also with running time $O(p(t)3^t)$ for deciding if a knot is the unknot, where t is the number of tetrahedra in an ideal triangulation of the knot complement. Note that there is a bound $t \leq cn$, where n is the crossing number of a projection of the knot and c is a (small) constant. We discuss this in detail elsewhere. Note that these algorithms avoid the computationally more expensive issue of deciding whether a given surface is incompressible.

Our other main algorithm is to determine if a given 3-manifold has an embedded incompressible surface or not. If the manifold is known to be irreducible (by applying our first algorithm), then this is the same as determining if it is Haken or not. As Thurston's uniformisation theorem applies to the class of Haken 3-manifolds, this is a key algorithmic issue in 3-manifold theory. In particular, few examples are known of non-Haken 3-manifolds and we hope that this algorithm will be useful for finding new ones.

This algorithm has running time $O(k^t)$, where k is a constant. We will give a rough upper bound on k and in another paper discuss some lower bounds for various important quantities involved in normal and almost normal surface theory.

A.Casson gave inspirational lectures at Montreal in 1995 and at the Technion in 1999 on related topics. In particular he outlined an approach to the problem of finding the connected sum decomposition in the latter talk and introduced linear programming as a key tool. He also described crushing normal surfaces in the former talk, as a way of simplifying triangulations. We will discuss his method and compare it to ours.

1. Introduction

Normal surface theory was introduced by Kneser [16] and developed by Haken [4], [5], [6] with a view to solving algorithmic questions in 3-manifold topology. In particular, Haken gave a solution of the problem of deciding if a knot in $\mathbf{R}^3$ was the unknot or not. Later, work of Hemion [8] completed the solution of the recognition or homeomorphism problem for Haken 3-manifolds, i.e irreducible 3-manifolds which contain embedded incompressible

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surfaces, modulo the 3-sphere recognition problem. So one can decide, given a pair of 3-manifolds, knowing that one is Haken, whether the two are identical (homeomorphic) or not. Rubinstein [19], [20],[21] and Thompson [22] gave a solution of the 3-sphere recognition problem, introducing the idea of almost normal surfaces. The first running time analysis of 3-manifold algorithms was given by Hass, Lagarias and Pippenger [7] in 1998. We will begin by quickly reviewing the theories of normal and almost normal surface theory.

In [15] and [12] it is shown that the essential spheres and tori involved in the connected sum and JSJ decompositions can be found at the vertices of the projective solution space of normal surfaces. However the number of such vertices grows exponentially (see [3] for some explicit bounds) and deciding whether surfaces are essential also involves exponential algorithms. Our aim in this paper is to outline better computational methods for solving these problems.

A key idea is to modify our triangulations at each step of the procedure. Since the number of tetrahedra is always the exponent in our complexity bounds, reducing this significantly speeds up successive steps of the algorithms. We employ the method of crushing, as described in [13], [14]. In particular, we will always start with a one-vertex triangulation of a closed 3-manifold or an ideal triangulation of the interior of a compact 3-manifold. As each desired normal surface is found, the manifold is split open along the surface and the boundary components crushed to points. Further crushing then occurs to reconstitute a triangulation of the pieces. For the first two algorithms of finding connected sum and JSJ decompositions, determining which pieces are essential is postponed to the final step of the process. In particular, we avoid directly deciding whether tori are incompressible.

For the final algorithm of deciding if there is any embedded incompressible surface, we introduce a 'sieve' type method. All normal surfaces with the same quadrilateral types are considered together, as they all have the same 'guts' when the manifold is split open along the surface. We then show that there is a simple method of deciding whether there is a compressing disk for such a surface, by looking only at the guts. (Note the guts may need to be modified during the process). Moreover, surfaces in 3-manifolds with fewer tetrahedra may have isomorphic guts, so there is a natural way of building a data-base of guts to check for incompressibility. Note that in [11] it was shown that if there is an embedded incompressible surface, then there is one at a vertex of the projective solution space. However this gives a doubly exponential algorithm since it takes exponential time to check each of the exponentially many vertices for incompressibility.

The methods here apply to both closed manifolds and ideal triangulations of the interiors of compact manifolds with boundary. We will concentrate on the closed case and just indicate some of the modifications required for ideal triangulations. A detailed version of these techniques will appear elsewhere. One especially interesting application of ideal triangulations is to knot and link complements in S^3. For that case, the connected sum decomposition corresponds to the factorisation of knots and links into prime summands along essential annuli. The JSJ decomposition corresponds to the decomposition into satellite or companion knots and links along essential tori. The final algorithm can be used to decide if a knot or link complement is 'small' i.e whether or not there is an embedded incompressible surface in the complement, which is not a boundary parallel torus.

We will also sketch a new solution to the knot triviality problem in section 3. In [4], Haken gave an algorithm by triangulating the complement of the knot and searching for an embedded normal disk. Haken's algorithm has the computational disadvantage that the triangulation has quite a lot of tetrahedra, since a regular neighbourhood of the knot is removed and the boundary torus appears as faces of the triangulation. In our approach, the knot is collapsed to an ideal vertex and ideal triangulations are easily produced, e.g by Jeff Week's program SNAPPEA. These have considerably fewer tetrahedra. The method is then to search for an almost normal torus in this triangulation and has running time $O(p(t)3^t)$, where t is the number of tetrahedra in the ideal triangulation.

For basic 3-manifold theory - see either [9] or [10]. For many properties of triangulations and normal surface theory - see [13].

2. Connected sum decompositions

We start with a very quick review of normal and almost normal surface theory. Given a triangulated 3-manifold, a normal surface meets the tetrahedra in normal disks which are either triangles cutting off a vertex or quadrilaterals, splitting a simplex into two triangular prisms. There are 7 different such normal disk types in each tetrahedron. The *coordinates* of a normal surface are then the $7t$ non-negative integers $(n_1, ..., n_{7t})$, where n_i gives the number of a given disk type and t is the number of tetrahedra. There are also $6t$ compatibility equations of the form $n_i + n_j = n_k + n_p$ where n_i, n_j are the number of triangular and quadrilateral disks in a given tetrahedron, with a given normal arc type in their boundary. Such an arc type is specified by the pair of edges of the triangulation which contain the end points of the arc. Then n_k, n_p are the number of triangular and quadrilateral disks of the adjacent tetrahedron with a common face, also containing this arc type. The solution space V is the linear subspace of vectors satisfying these equations. V meets the positive octant of $\mathbf{R}^{7t}$ in a cone. Intersecting with the hyperplane $\sum_{1 \le i \le 7t} n_i = 1$ gives the projective solution space $\mathcal{P}$; a compact convex polytope. The results in [11], [15], [12] show that most 'interesting' surfaces in M can be found at the vertices of $\mathcal{P}$. This will be critical to us, as one can use linear programming (LP) to run through the set of vertices, using a linear objective function, in polynomial time. Our polynomial times exponential bounds will occur, since exponentially many such LP procedures will be needed.

It is possible to use different coordinate systems, particularly canonical and quadrilateral coordinates - see e.g [23], [21]. It turns out that 'forgetting' the triangular coordinates, loses one degree of freedom in the solution space, for one vertex triangulations. So projecting from V to W, the solution space in quadrilateral space $\mathbf{R}^{3t}$, only reduces dimension by one (see [23]). The normal surface generating the kernel of this projection is the vertex linking 2-sphere S_v, i.e the vector consisting of one of each of the triangular normal disk types. We will use both standard normal coordinates (the n_i) and quadrilateral coordinates for the outline of the algorithms in this paper.

We now describe the algorithm to find the connected sum decomposition of a closed orientable 3-manifold M, then indicate how to proceed for interiors of compact manifolds with boundary. The idea is to split M along disjoint embedded 2-spheres into pieces, none of which are a ball with smaller open balls removed. If the boundary 2-spheres are then filled in with balls, we get the connected summands of the decomposition. If only separating 2-spheres are used, some of these summands can be copies of $S^1 \times S^2$. Any summand different from $S^1 \times S^2$ must be irreducible, i.e has the property that any embedded 2-sphere bounds a 3-ball. The decomposition is unique, i.e independent of the way that the 2-spheres are chosen. The existence of such a decomposition was proved by Kneser and the uniqueness by Milnor.

The idea, following e.g [15], is to search for an embedded normal 2-sphere which gives either a connected sum decomposition or $S^2 \times S^1$ factor of M. So in [15], it is proven that such a sphere can be found at a vertex of the projective solution space. It is not difficult to decide whether a sphere S is separating or not - if M is split open along S, one just has to see if there are one or two connected components of the resulting manifold. In the case of a single component, finding an arc λ from one side of S to the other side then becomes easy. A regular neighbourhood of $S \bigcup \lambda$ is a copy of $S^2 \times S^1$ with an open ball removed.

Now we sketch how to find S at a vertex of $\mathcal{P}$ and then discuss the case that S separates M into two components. In particular, we need to deal with the case that one of these components is a ball, i.e we have found a trivial sphere and do not want to use this in our connected sum decomposition. This also is a good point at which to introduce the idea of 0-efficiency of the triangulation as discussed in [13]. We will denote the given triangulation by $\mathcal{T}$; a key idea will be to modify the triangulation as the algorithm progresses.

A triangulation is called 0-efficient if the only embedded normal 2-sphere is vertex linking, i.e the boundary of a small regular neighbourhood of some vertex. In [13], we show that any triangulation of a closed orientable irreducible 3-manifold, different from $S^3, RP^3, L(3,1)$ can be modified to be 0-efficient. Moreover, it is proved there that either there is only one vertex in a 0-efficient triangulation $\mathcal{T}$ or the manifold is homeomorphic to S^3 and there are precisely 2 vertices. We will only be concerned with the first case, so can restrict to one vertex triangulations. The first step in [13] is to convert any triangulation of the original manifold M to be one vertex. This is achieved by engulfing all the vertices of the triangulation in a ball B^3 bounded by a normal sphere S. Finding such a sphere can be done by choosing the boundary of a small neighbourhood of a maximal tree in the one skeleton of $\mathcal{T}$ and pushing this outwards to form S. So the ball bounded by S can be found by a fast algorithm and is then crushed to a point. It is shown that the resulting manifold is again homeomorphic to M and has a new triangulation with fewer tetrahedra, unless the original manifold was S^3 or had a connected sum factor either RP^3 or $L(3,1)$. Moreover, the method in [13] is also a quick algorithm, with possibly further crushing to get the new triangulation. So we can assume to start that $\mathcal{T}$ already has only one vertex. (One slight complexity is that in some cases, one might end up crushing a region which is a 3-ball with some smaller open 3-balls removed, rather than just a 3-ball. This splits M into a number of pieces but we have a very similar situation in the next step so will not discuss this possibility separately).

Euler characteristic χ is employed to run through the vertices of some faces of $\mathcal{P}$ to find a non vertex linking sphere S, if one exists. The first point is to consider how to formally define the Euler characteristic on $\mathcal{P}$. For each vertex solution V, one can multiply its rational coordinates to form the smallest non-negative integer solution. We will denote this integer vector by V again. It is easy to show that a corresponding normal surface must be connected, since otherwise V would not be a vertex solution. Moreover, an embedded surface is obtained if and only if in each tetrahedron, there is at most one non-zero quadrilateral coordinate. So it is easy to check whether a given vertex V could give an embedded (connected) surface.

To define χ, for each triangular or quadrilateral disk D or D', we define $\chi(D) = \frac{1}{2} - \sum_{1 \le i \le 3} \frac{1}{d_i}$ and $\chi(D') = 1 - \sum_{1 \le i \le 4} \frac{1}{d_i}$, where the degrees d_i are of the edges at the corners of D or D'. We then define the Euler characteristic of V as the appropriate sum of values of χ on the disk types.

Now χ is a linear functional on the integer vectors $\mathbf{Z}^{7t} \subset \mathbf{R}^{7t}$. Moreover the maximum value of χ for connected surfaces is clearly 2, which occurs uniquely at normal spheres. At a particular vertex V, a neighbouring vertex V' (ie the two vertices are joined by an edge in $\mathcal{P}$) can always be found with larger value of χ unless $\chi(V)$ is already a maximum.

The final step is to show how to avoid ending at the vertex linking sphere S_v or at some non embedded (i.e singular) sphere. We want to restrict to faces of $\mathcal{P}$ containing only embedded surfaces and not containing S_v. This can be done conveniently by first working in quadrilateral space. Actually, since quadrilateral coordinates determine standard coordinates, up to addition of copies of S_v, it turns out there are at most 3^t possibilities for such faces. To see this, note that a maximal face of embedded surface vectors in quadrilateral coordinates is determined by choosing one quadrilateral coordinate possibly non zero in each tetrahedron. Each of the $O(3^t)$ choices can then be pulled back uniquely to a face in $\mathcal{P}$, avoiding S_v.

Using LP to run through the vertices of each such face, we can determine if there is an embedded normal 2-sphere which is not vertex linking. If we find such a sphere S which separates M into two components M_1, M_2, we crush the copy of S in each component to a point. By further crushing as in [13], either a new triangulation with fewer simplices of the pieces M_1, M_2 is found, or a connected summand RP^3 or $L(3,1)$ is split off. Another way of viewing this process is to enlarge the normal 2-sphere S to obtain a submanifold N of M which is a ball with some smaller open balls removed, so that all the boundary components are normal 2-spheres. So the simplest case is just a small regular neighbourhood of S. However in some cases in [13], one must add a disk to S and then thicken up the resulting 2-complex to obtain N. One can then split M open by removing the interior of N and crush the boundary spheres of the complementary components to points.

The algorithm can be iterated until 0-efficient triangulations of pieces of M are found. The last step is to decide which of these pieces are non trivial summands, i.e to run the 3-sphere recognition algorithm. Let M now denote one of these pieces, with a one vertex 0-efficient triangulation. As in [21], [22], M is homeomorphic to S^3 if and only if there is an almost normal embedded 2-sphere S', with a single octagonal piece. Assuming there are t tetrahedra in the triangulation, there are $3t$ possible choices of this octagon.

Construct a new almost normal space, consisting of surfaces which are built from triangles and quadrilaterals as usual and also multiples of one fixed choice of octagon. As before, there is a projective solution space $\mathcal{P}'$ of such surfaces. It is straightforward to show that if there is an almost normal 2-sphere, then there is one at a vertex of $\mathcal{P}'$. However it is not obvious that this 2-sphere has a single octagon, rather than several parallel ones. Now we give two ways of finishing the argument. Casson in 1999 used the functional $\tau = \chi - n$, where n is the number of octagons, on the vertex surfaces of $\mathcal{P}'$. An almost normal 2-sphere with exactly one octagon is the only surface with positive value of τ, different from the vertex linking 2-sphere. So Casson finds such a surface by performing LP to maximise τ on each maximal face of embedded surfaces (in quadrilateral and octagonal coordinates, pulled back to avoid S_v) of each of the $3t$ almost normal projective solution spaces. So again the running time is $O(p(t)3^t)$.

Our approach is required in the next section, since Casson's trick does not work for tori. We give a quick summary of this alternative method to find the almost normal 2-sphere, if it exists. Take χ instead as the functional. So as before, using LP, a maximum value of χ can be found on the vertices of each of the maximal faces. This will give a sphere S' which cannot be normal (as the triangulation is 0-efficient and the faces are chosen to not contain S_v) but might have k parallel octagons. If k is odd, the same method as in [13] or [21] or [22] shows we can push the sphere off to both sides and shrink it to a point. So the manifold piece under consideration must be S^3. If k is even, we can split M along S' into new pieces M_1, M_2. Notice on one side of S', say M_1, we see parallel regions between octagons. If the boundary copy of S' is crushed to a point in M_1, we can crush all these product octagonal regions to intervals and so can complete the same process as in [13] to obtain a new triangulation. On the other hand, S' can be pushed off into M_2 and shrinks to a point, showing that M_2 is homeomorphic to a ball. So this enables us to produce a new smaller triangulation of M and repeat the process. It terminates after a finite number of steps and gives a different way of implementing the S^3 recognition algorithm. The running time is also $O(p(t)3^t)$, although it may require more splittings and crushings than Casson's method.

An important remark about the iteration step is as follows. We would like to avoid excessive iterations, which might conceivably increase the running time bound of the first step $O(p(t)3^t)$. Notice as in [15] or [12], a maximal collection of disjoint normal and almost normal 2-spheres can be found by each of our LP searches in a maximal embedded face. Kneser's lemma ([16]) gives a linear bound on the number of such surfaces. So we don't actually have to perform an iteration, rather a two step process is enough, since the first step must give the connected sum decomposition amongst one of the maximal collections ([12]).

To complete this section, we make some comments about ideal triangulations. Decomposition along spheres follows a very similar pattern to that above. Suppose we want to decompose along essential annuli (as in the knot factorisation problem). First blow up the triangulation (cf [14]) to a one vertex triangulation or truncate the ideal tetrahedra and subdivide into new tetrahedra. Next search for essential annuli at the vertices of the projective solution space of proper normal surfaces. Only surfaces with meridinal slope are considered. An Euler characteristic LP argument again works to find such annuli. We now have a more complicated problem to decide if such an annulus is essential. If the annulus is boundary parallel, it will split off a solid torus region. In the next section, we address how to determine this efficiently, without checking compressibility of the boundary torus of the region. Finally, if the annulus boundary compresses to an essential compressing disk, then the original knot is the unknot. We also discuss this problem in the next section.

Remark

In [12], it is proved that all the connecting spheres for a connected sum decomposition of M can be found in a single face of $\mathcal{P}$. However the algorithm above may be better in practice, since it searches through the faces of $\mathcal{P}$ for collections of disjoint normal 2-spheres which are not vertex linking and then uses them to simplify the triangulation. Even if the correct face of $\mathcal{P}$ could be somehow identified, one still has the problem that some of the spheres at the vertices of this face might bound 3-cells or cobound a ball with smaller open balls removed. So these issues must be checked using the 3-sphere recognition algorithm at any rate.

3. JSJ decompositions

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This section is rather similar to the previous one, except it depends on the more complicated theory of 1-efficient triangulations in [14]. Jaco-Shalen and Johannson independently gave a decomposition of a closed orientable irreducible 3-manifold M along a collection of disjoint incompressible embedded tori into pieces which are Seifert fibred or atoroidal, i.e any embedded incompressible torus in a complementary piece is parallel into the boundary and there are no proper essential annuli. We refer to this as the JSJ decomposition and a reference is [10]. It turns out that if the decomposition is non empty, i.e there is at least one such torus, then the pieces are Seifert fibred or have canonical complete hyperbolic metrics of finite volume, by Thurston's uniformisation theorem. A similar result works for manifolds with incompressible boundary tori, as for instance for the complement of a small neighbourhood of a non trivial knot or link in S^3.

We sketch quickly an algorithm, with running time $O(p(t)3^t)$ to find the JSJ decomposition of a closed orientable irreducible 3-manifold M and also discuss the bounded case. Using the ideas of the previous section, we may suppose that M comes equipped with a 0-efficient one vertex triangulation $\mathcal{T}$. The aim is to search for embedded normal tori which are not the standard ones, analogous to the vertex linking sphere S_v. We will only give an indication of the types of tori under consideration and refer the reader to [14] for full details.

The first type of standard normal torus, is a thin edge linking one. Each edge of $\mathcal{T}$ is a loop since there is only one vertex. If a small regular neighbourhood of such a loop is a normal torus, we call this thin edge linking. The next type is a thick edge (linking) torus. In this case, one can expand out the neighbourhood of an edge loop in a standard way to form a solid torus which is called layered. The first tetrahedron containing the edge loop has two faces identified together to form a Mobius band. This gives a one tetrahedron triangulation of a solid torus and is the simplest thick edge torus. Subsequent tetrahedra are glued on to the boundary torus along two faces. The Euclidean algorithm for finding greatest common divisors gives an elegant way of labelling the tetrahedra of such triangulated solid tori, which form layered triangulations.

The final type of solid tori are called flat. These are very special examples and arise in situations where a normal torus in a triangulation bounds a solid torus with only one tetrahedron 'inside' i.e with the truncated tetrahedron inside the solid torus. All other tetrahedra of the triangulation meet the outside of the solid torus in a truncated tetrahedron or truncated prism.

We define a 1-efficient triangulation. Such a triangulation is characterised by the requirements that any embedded normal torus bounds a solid torus and is either thin edge linking, thick or flat. Moreover the collection of thick and flat tori form a 'wedge' in the sense that the solid tori they bound can be shrunk slightly to meet only at the vertex. The outside of this wedge is called the core of the manifold. The main result of [14] is that for any closed orientable irreducible atoroidal 3-manifold M which is not a small Seifert fibred space, any triangulation can be modified to be 1-efficient by moves which decrease the number of tetrahedra in the core.

A brief idea is now given of how the procedure in [14] works. Given an initial triangulation $\mathcal{T}$, we can suppose it is one vertex and 0-efficient. Now if there is any embedded normal torus T which is not one of our three standard types, it can be shown that T bounds a solid torus. This uses the atoroidal condition, which implies that T is compressible. The case that T bounds a region which is a cube with knotted hole is ruled out, since then a barrier argument implies that $\mathcal{T}$ has a non vertex linking normal sphere surrounding this region.

The next step is to remove the interior of the solid torus and collapse the boundary torus to a point, creating an ideal triangulation of the solid torus complement. (At this stage, we need to rule out small Seifert fibred spaces, where the whole triangulation might collapse). As in the previous section, actually more of the triangulation might collapse, so we could enlarge the solid torus first, as an equivalent way of proceeding.

The ideal triangulation is then 'blown up' by adding back in tetrahedra in a controlled way to create a triangulation with one vertex on the boundary. So the boundary consists of two triangles forming the original torus T. We complete the process by adding in a layered triangulation of the solid torus. The whole process must be done so that the number of tetrahedra in the core decreases. If a flat torus was subjected to this process, the number of tetrahedra in the core might increase, so we have to leave such flat solid tori unchanged.

Finally the process can be iterated. At each stage, removing an open solid torus and crushing the boundary to a point may destroy previous thick solid tori or they may be preserved. In the latter case, a larger wedge of layered solid tori (and possibly flat solid tori) is removed to form the core and the process must terminate after a finite number of steps.

To find the JSJ decomposition is intimately connected to the algorithm to convert a triangulation to a 1-efficient one. Suppose that we have a 0-efficient one vertex triangulation of a closed orientable irreducible 3-manifold M. The first step is to search for normal tori which are not edge linking, thick or flat. The idea is very similar to the search for non vertex linking 2-spheres. We will use the Euler characteristic χ as a linear functional on maximal faces of embedded surfaces in $\mathcal{P}$. These faces are defined as in the previous section by pulling back faces from quadrilateral space to avoid the vertex linking 2-sphere S_v. Consequently, χ will be maximised at normal tori. The problem is to avoid the standard ones.

Notice that there are a very small number (less than $2t$) of standard thin edge linking, thick or flat normal tori and we can construct all of these quickly knowing the triangulation $\mathcal{T}$. Since we can define χ as a linear functional on each maximal face, the collection of vertex normal tori and edges joining them in $\mathcal{P}$ form a connected graph Γ. (We are ignoring the possibility of Klein bottles, which are easy to handle also). So it is now straightforward to decide, assuming the LP yields a standard normal torus at a vertex, whether there are some non standard normal vertex tori, by searching amongst adjacent vertices in $\mathcal{P}$, hence following along Γ. By e.g [15] , we know that if there is a non trivial JSJ decomposition, then the incompressible tori can be found at vertices of $\mathcal{P}$. As usual, this step involves checking at most 3^t faces and so the running time of this step is $O(p(t)3^t)$.

We now must combine this approach with the techniques outlined previously for constructing 1-efficient triangulations. Namely, split M open along a separating embedded normal torus T. (It is easy to check that any non separating embedded normal torus must be incompressible, since M is irreducible. We also split M open along T in this case.) Next, collapse the boundary copies of T to a point, constructing an ideal triangulation of the complementary pieces. If one of the complementary pieces is Seifert fibred, we may find the triangulation completely collapses, but this is easy to detect and identifies the piece completely.

Iterate the construction, using the new ideal triangulation. We need to run the algorithm of the previous section, to ensure the ideal triangulation is 0-efficient. Next, search for other embedded normal tori. For ideal triangulations, there are no standard tori except for the vertex linking ones, so this simplifies matters, so finding tori is more like the sphere case. Each time an embedded normal torus is found, the manifold is split open and the boundary tori are collapsed to ideal vertices. At some stage the process stops, since the number of tetrahedra in the individual pieces is clearly decreasing. (Note by making sure the triangulation is 0-efficient, a normal torus bounding a cube with knotted hole cannot occur, as discussed previously.)

So the end result is that the pieces have ideal 0-efficient triangulations and there are no embedded normal tori which are not vertex linking. To complete the algorithm, we need to decide if any piece is a solid torus, a product of an interval and a solid torus, or a product of a pair of pants and a circle. For any other product of a circle and a punctured 2-sphere would have non boundary parallel embedded normal tori. In the first two cases, it is straightforward to show that the sweepout or thin position argument ([21], [22]) gives an almost normal embedded torus. We can find this by a similar argument to the S^3 recognition argument outlined in the previous section.

Firstly, we want to indicate why such an almost normal torus can be found at a vertex of almost normal surface space. Since the ideal triangulation is 0-efficient, there are no embedded normal (or almost normal, meaning several parallel octagons) 2-spheres. So writing a positive multiple of an almost normal torus as a sum of positive multiples of vertex surfaces, using Euler characteristic we see that all the vertices in this sum must also be tori. So some of these are almost normal as required. (Normal and almost normal projective planes and Klein bottles can be easily dealt with.)

Note, to locate the vertex almost normal tori, we have to use the argument involving the functional χ, since Casson's functional cannot be applied here. So one constructs an almost normal projective solution space $\mathcal{P}$ for each choice of octagon. By pulling back maximal faces of embedded surfaces from quadrilateral and octagon space, we get $O(3^t)$ faces not containing any vertex linking tori. Using χ as the functional for LP will find any non vertex linking tori in almost normal space. If such a torus T has an odd number of octagons, we conclude by a push off and shrinking argument, that the manifold is either a solid torus or product of a torus and an interval. For an even number, one side must be a solid torus or product of torus and interval, and for the other side we can collapse the boundary copy of T to a point, since all regions on this side which are products of octagons and intervals can be collapsed to intervals. So we can repeat the process till it terminates.

To deal with regions which are products of a pair of pants and a circle, it is easy to truncate, retriangulate and search for an embedded essential annulus by a similar method. (One works with the projective solution space of proper normal surfaces. Again excluding disks, spheres and tori means that annuli maximise χ.) Alternatively, we can blow up the ideal vertices to one vertex triangulated boundary tori as in [14]. The advantage of this latter techique is that the number of tetrahedra is not larger than the number in the original manifold. So we conclude that the original running time bound of $O(p(t)3^t)$ is still valid.

Notice that if we find that one of the complementary regions is a solid torus, then we need to reinsert this piece and start again. The way to do this is to perform triangulated Dehn surgery as in [14]. Namely, the ideal triangulation is not 1-efficient and we can reduce the number of tetrahedra, by collapsing this solid torus to a point, then blow up the ideal vertex to a torus, fill in by a layered or thin edge linking solid torus, finally collapsing an edge between an ideal vertex and the new non ideal vertex. (Obviously the last step involves crushing all the tetrahedra adjacent to this edge). This has the effect of decreasing the number of tetrahedra in the ideal triangulation. We can therefore assume that no piece is a solid torus.

Analogously to the previous section, by choosing maximal collections of disjoint normal (and almost normal) tori in the maximal faces of embedded surfaces, a two step process is sufficient, since one of these maximal collections will contain the JSJ splitting tori. So the iteration does not increase the running time bound of $O(p(t)3^t)$. This is actually more subtle than before, since we have an additional possibility of an embedded normal solid torus lying in a knotted way inside a larger solid torus. Note here that the larger normal solid torus has an associated normal or almost normal torus (by sweepouts or thin position) which must meet the smaller knotted normal solid torus, by an easy barrier argument. So a search for maximal disjoint collections including both normal and almost normal tori enables us to deduce this situation in the initial step, avoiding more iterations.

Searching for annuli is also required to decide which of the complementary pieces of the JSJ decomposition are Seifert fibred and which are hyperbolic. Note that in [18], there is a theoretical algorithm to decide if a closed orientable irreducible 3-manifold is a small Seifert fibred space, i.e is Seifert fibred but does not have any embedded incompressible tori. This has a rather large running time bound. Our bound only works under the assumption that the 3-manifold is Haken, so for example if it is Seifert fibred, then it does contain embedded incompressible tori.

Clearly the method discussed works equally well for ideal and one vertex triangulations. So we can find JSJ decompositions of knot and link complements, i.e the companion or satellite factorisations.

Remark

Finding an efficient algorithm to decide if a knot is the unknot is a particularly interesting question. Notice that the method in this section gives such an algorithm, with running time $O(p(t)3^t)$, where t is the number of tetrahedra in an ideal triangulation of the knot complement. We will discuss bounds for t in terms of a projection of the knot elsewhere.

To summarise, given an ideal triangulation $\mathcal{T}$ of the knot complement in S^3, we want to decide if this manifold is an open solid torus. So we first search for embedded normal tori and split along them into pieces with new ideal triangulations, by collapsing the boundary tori to points. So the number of tetrahedra in the triangulations of the pieces decrease rapidly with each step. Now once the process terminates, check for almost normal tori in the pieces. Unless all the pieces glue together to form a solid torus, we conclude that the knot is not the unknot.

So this also completes the brief discussion at the end of the previous section on the algorithm to find the prime factorisation of a knot.

4. Finding incompressible surfaces

In this section, we outline a method for deciding if a given closed orientable irreducible atoroidal manifold M, with a 1-efficient triangulation $\mathcal{T}$ has an embedded incompressible surface or not. By [11], we know that if there is such a surface, one can be found at a vertex of the projective solution space $\mathcal{P}$. However the direct approach then gives a doubly exponential algorithm, since there can be exponentially many vertex solutions of exponential size and for each, one must split the manifold open and retriangulate, followed by a search for an embedded compressing disk in one of the complementary pieces.

Our idea is to run through all the vertex solutions quadrilateral space corresponding to embedded surfaces, as a possible source of incompressible surfaces. It is straightforward to show that all vertices in $\mathcal{P}$ corresponding to embedded surfaces project to vertices in quadrilateral space. So no information is lost this way. Now it is easy to generate these vertex solutions efficiently and the difficulty is to avoid splitting open and retriangulating for the exponentially large surfaces which can arise.

The key idea is to examine the guts of the complementary regions of an embedded normal surface S, which is a vertex in quadrilateral space. If S is non separating, then M has infinite first homology and it is well-known that M has a non separating incompressible surface and is Haken. So this case is not interesting. Note that each vertex is described by a choice of at most one quadrilateral type in each tetrahedron; giving the bound 3^t of the number of such vertices. If a tetrahedron contains a non zero quadrilateral type of S, we split it into two truncated prisms using this quadrilateral together with all four triangular normal disk types. It is convenient to slightly shrink each truncated prism by slightly pushing in all of its quadrilateral faces. For all tetrahedra which do not contain any non zero quadrilateral type of S, we truncate using the four triangular disks. Then glue together the truncated prisms and tetrahedra to yield pieces $P_1, P_2, .., P_k$ of the complementary regions of S. The gluings occur only along hexagonal faces, leaving the quadrilateral faces free. Note the original truncated prisms meet faces of $\mathcal{T}$ in hexagons and quadrilaterals - the latter have been slightly pushed into the tetrahedra. We get annuli $A_1, A_2, .., A_m$ formed by all these latter quadrilaterals. The triangular faces of the truncated prisms and tetrahedra are part of either the vertex linking sphere S_v or S and the remaining quadrilaterals faces are all part of S. We call $P_1, P_2, .., P_k$ the guts of the surface S in M. The boundary of the guts therefore consists of regions in S_v or S together with the annuli $A_1, A_2, .., A_m$.

We will use the guts to simplify the problem of determining whether S is compressible or incompressible. So we need to understand how a compressing disk for S can meet the guts. If we remove the small open ball B_v around the vertex v bounded by S_v, then the complementary regions M_1, M_2 for S in $M \setminus B_v$ can be conveniently described by gluing product regions to the guts $P_1, P_2, .., P_k$ along the annuli $A_1, A_2, .., A_m$. The product regions are I bundles over subsurfaces of S or S_v and are unions of products of intervals and triangular or quadrilateral disk types in the tetrahedra.

The annuli $A_1, A_2, .., A_m$ are properly embedded in M_1, M_2. Assume first they are all incompressible and non boundary parallel. Then it is an easy cut and paste argument to prove that S is compressible if and only if there is a compressing disk D in one of the components P_i of the guts, with ∂D in $\partial P_i \cap S$. The main observation is that the parts of M_1, M_2 outside the guts are all I bundles over subsurfaces of S or S_v. So there cannot be a compressing disk outside of the guts, unless one of the annuli is actually compressible there. Moreover as the annuli are supposed to be incompressible and boundary incompressible, we can shift any compressing disk for S off all the annuli. Now it is straightforward to decide if such a compressing disk occurs inside the guts. In fact, any truncated tetrahedron can be split into 9 tetrahedra. So a bad bound is $9t$ simplices for a triangulation of the guts. Hence a compressing disk in the guts can be found in running time $O(3^{9t})$. A more efficient technique is to crush the boundary of the guts to a point and then to blow up the resulting ideal vertex, in the manner of [14]. We will discuss the resulting bound elsewhere.

Note that if one of the annuli A_i is boundary compressible inside the guts, then either S is compressible inside the guts (the result of a boundary compression of an annulus is a disk) or A_i is boundary parallel and a piece P_j of the guts is a solid torus. If A_i is compressible in a component P_j of the guts then there is a compressing disk for S in P_j obtained by sliding the disk off A_i onto S. However such a compressing disk for S in P_j (and similarly a disk found from a boundary compression of A_i) might be trivial, i.e parallel into S. It is easy to see this can only happen if A_i is also compressible in the adjacent outside I bundle region, which is then a product of a disk and an interval. We call such a product region a plug for the annulus A_i. So the conclusion is that in all cases, either there is a non trivial compressing disk for S inside some piece P_j of the guts, or some piece P_j is a solid torus or there is a plug for some annulus A_i.

To complete the argument, we need to discuss the two cases above when a piece P_j of the guts is a solid torus and when a product region is a plug for an annulus A_i. In the first case, ∂P_j consists of two parallel annuli, one in S or S_v and the other A_i. In this case, it is easy to see that for S to be compressible, there must be either a plug for another annulus A_m or a non trivial compressing disk in a different piece of the guts P_q. So it suffices to deal with the case of a plug.

To detect a plug, we follow a recent method of Agol, Hass, Thurston ([1]). Namely, we construct the surface S and remove all the interiors of the regions $S \cap \partial P_j$ where the surface intersects the guts. So a bounded number of components of the intersection of S with the I bundle regions is formed. By [1], there is a polynomial time algorithm to decide on the topological type of these components, even though the surface S may have an exponential number of pieces. So we can find the Euler characteristic of these pieces and so decide which ones are plugs. Once such a plug for an annulus A_i has been detected, the easiest procedure is to collapse the A_i to an interval, thus pinching part of ∂P_j, where P_j is the piece of the guts containing A_i. Notice that this changes the topological type of P_j. Moreover additional collapsing of the polyhedra forming P_j must be carried out, in a very similar manner to [14]. Let P_j' denote the new guts piece formed this way, so that it also has a decomposition into truncated prisms and truncated tetrahedra. We can then repeat the search for a compressing disk in the new guts piece P_j'. The process can be iterated at most as many times as the number of annuli and the complexity goes down at each stage, since clearly P_j' has fewer polyhedra than does P_j.

It is possible that a piece of the guts might get engulfed in a plug as in the following situation. Suppose that some guts component P_j is a solid torus and has the structure of the product of an interval and an annulus. Assume also that one of the four boundary annuli of P_j is an A_i which bounds a plug. Then the union of P_j and the plug can be thought of as a larger plug to be added to the I bundle region adjacent to the second annulus A_k, which must be in ∂P_j. So we can form a larger I bundle region and continue the procedure.

If at any stage we find no additional plugs or no compressing disks for the modified guts, then the conclusion is that our vertex surface S is indeed incompressible.

Remarks

1) To summarise; the algorithm to decide if there is an incompressible surface has running time $O(k^t)$, where $k = 4 \times 3^9 = 78,732$, since a rough bound is 4^t vertex surfaces to check for incompressibility, each taking time at most $O(3^{9t})$. We can improve this bound by estimating more carefully the size of the guts and also triangulating by a crushing and blow up procedure. Morever, since the ideal triangulation of the pieces of the guts with boundary crushed will have rather few tetrahedra, we would expect to often see the same triangulation as before, or at least a large intersection. So a sieve approach, building a data base of small ideal triangulations of manifolds with incompressible or compressible surfaces at the link of the ideal vertex, may considerably speed up the process.

2) We extend these ideas elsewhere to give algorithms to find very short hierarchies ([2], [21]) and to decide if two Haken manifolds are homeomorphic or not, both with running time k^t with constant k. A very short hierarchy is a system of incompressible surfaces of length 3 required to cut a Haken manifold up into cells. So one has a collection of disjoint closed incompressible surfaces at the first level, a system of disjoint incompressible surfaces with boundary (spanning surfaces) at the second level and then compressing disks at the third level. Such hierarchies were introduced in [2] and were also known to Gabai. Very short hierarchies are most convenient for the recognition algorithm for Haken manifolds.

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